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Dynamics of a bar of asymmetric cross-section

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Summary

An elastic bar of asymmetric cross-section is supported at two points and loaded by constant gravity force and axial force. The principal bending moments are assumed to be proportional to the principal curvatures, and the torque proportional to the twist. The equations of motion reduce to the sine-Gordon equation. The solutions corresponding to one soliton and to the collision of two solitons are given.

1. Basic formulae

Consider an elastic bar of asymmetric cross-section (Fig. 1). In the fixed coordinate system (x, y, z) the position of each cross-section z is defined by two displacement components

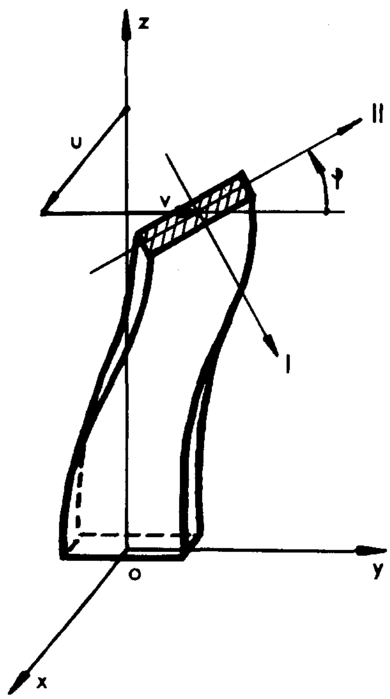

Figure 1. Coordinate system, displacements u, v and rotation angle ϕ of elastic bar.

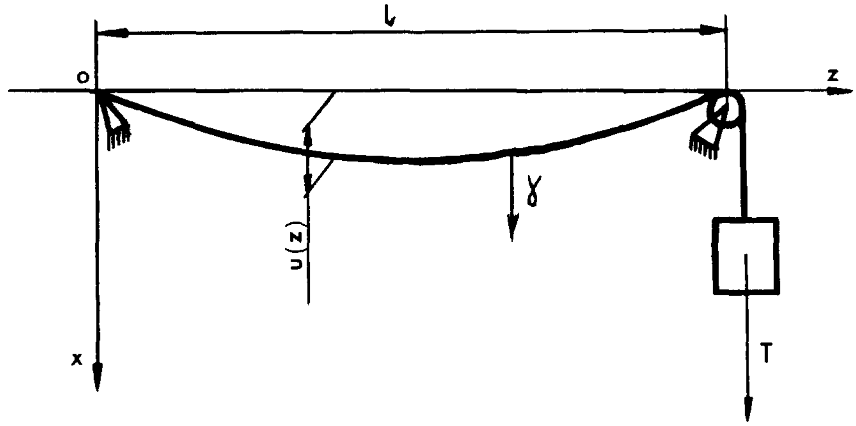

Figure 2. Forces acting on the bar, γ = gravity, T = axial force.

$u = u(z, t)$, $v = v(z, t)$ and the rotation angle $\varphi = \varphi(z, t)$. The bar is supported at two points $z = 0$ and $z = l$, where $u = v = 0$. The displacement is assumed to be small as compared with the distance l.

On the bar there acts the constant gravity force γ (per unit length) in the x-direction and the (almost) constant axial force T (Fig. 2). The bar possesses an infinite set of equivalent equilibrium positions corresponding to $\varphi = 0, 2\pi, 4\pi, \ldots$ and another set of (unstable) equilibrium positions corresponding to $\varphi = \pi, 3\pi, 5\pi, \ldots$. This fact suggests that along the string a solitary wave may propagate. Note that, besides these trivial equilibrium configurations, the bar posseses an infinite set of essentially different non-trivial equilibrium configurations corresponding to $\varphi = 0$ at $x = 0$ and $\varphi = \pi, 2\pi, 3\pi, \ldots$ at $x = l$.

A description of the solitary wave is already provided by an engineering approach. Assume that the axial deformation is negligible, and that the principal bending moments M_{I} and M_{II} are proportional to the principal curvatures κ_{II} and κ_{I},

$$M_{\mathrm{I}} = -EI_{\mathrm{I}}\kappa_{\mathrm{II}}, \qquad M_{\mathrm{II}} = EI_{\mathrm{II}}\kappa_{\mathrm{I}}, \tag{1.1}$$

where I_{I}, I_{II} are the principal moments of inertia and E is Young's modulus. The twisting moment M is assumed to be proportional to the twist

$$M = K\kappa, \quad K = \text{constant}. \tag{1.2}$$

Because the displacements are small, the calculations may be based on the approximate formulae

$$\kappa_x = \frac{\partial^2 u}{\partial z^2}, \qquad \kappa_y = \frac{\partial^2 v}{\partial z^2}, \qquad \kappa = \frac{\partial \varphi}{\partial z}, \tag{1.3}$$

$$\kappa_{\mathrm{I}} = \kappa_x \cos\varphi + \kappa_y \sin\varphi, \qquad \kappa_{\mathrm{II}} = -\kappa_x \sin\varphi + \kappa_y \cos\varphi. \tag{1.4}$$

2. Equations of motion

The equations of motion will be derived from the Lagrange function

$$L = E_p - E_k, \tag{2.1}$$

where E_k is the kinetic and E_p the potential energy of the bar. Denoting by m the mass density (per unit length) we have

$$E_k = \tfrac{1}{2}m(\dot{u}^2 + \dot{v}^2) + \tfrac{1}{2}m(I_{\mathrm{I}} + I_{\mathrm{II}})\dot{\varphi}^2, \tag{2.2}$$

$$E_p = \tfrac{1}{2}E\left(I_{\mathrm{I}}\kappa_{\mathrm{II}}^2 + I_{\mathrm{II}}\kappa_{\mathrm{I}}^2\right) + \tfrac{1}{2}K\kappa^2 - \gamma u + \tfrac{1}{2}T(u'^2 + v'^2). \tag{2.3}$$

The last two terms in E_p represent the energy of the external forces, γ and T. Differentiation with respect to time has been denoted by a dot and differentiation with respect to z by a prime. The formulae (1.1)–(1.4) lead to the following expression for L:

$$\begin{aligned} L = \tfrac{1}{2}E\left[I_{\mathrm{I}}(-u'' \sin\varphi + v'' \cos\varphi)^2 + I_{\mathrm{II}}(u'' \cos\varphi + v'' \sin\varphi)^2\right] \\ + \tfrac{1}{2}K\varphi'^2 - \gamma u + \tfrac{1}{2}T(u'^2 + v'^2) - \tfrac{1}{2}m(\dot{u}^2 + \dot{v}^2) - \tfrac{1}{2}m(I_{\mathrm{I}} + I_{\mathrm{II}})\dot{\varphi}^2. \end{aligned} \tag{2.4}$$

The Lagrange function L depends on the second derivatives of u, v, φ; therefore the equations of motion are

$$\frac{\partial L}{\partial q} - \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial}{\partial z}\frac{\partial L}{\partial q'} + \frac{\partial^2}{\partial t^2}\frac{\partial L}{\partial \ddot{q}} + \frac{\partial^2}{\partial z^2}\frac{\partial L}{\partial q''} = 0. \tag{2.5}$$

Taking in turn $q = u$, $q = v$, and $q = \varphi$, and neglecting the products of the derivatives of φ and u and of φ and v (e.g. $u'\dot{\varphi}$, $\dot{v}\ddot{\varphi}, \dots$) the following system of equations is obtained

$$\begin{aligned} &-2E(I_{\mathrm{I}} + I_{\mathrm{II}})u^{IV} + \tfrac{1}{2}E(I_{\mathrm{I}} - I_{\mathrm{II}})(u^{IV}\cos 2\varphi + v^{IV}\sin 2\varphi) + Tu'' + \gamma = m\ddot{u}, \\ &-2E(I_{\mathrm{I}} + I_{\mathrm{II}})v^{IV} + \tfrac{1}{2}E(I_{\mathrm{I}} - I_{\mathrm{II}})(u^{IV}\sin 2\varphi - v^{IV}\cos 2\varphi) + Tv'' = m\ddot{v}, \\ &E(I_{\mathrm{I}} - I_{\mathrm{II}})(u''\cos\varphi + v''\sin\varphi)(-u''\sin\varphi + v''\cos\varphi) + K\varphi'' = m(I_{\mathrm{I}} + I_{\mathrm{II}})\ddot{\varphi}. \end{aligned} \tag{2.6}$$

Note that

$$u = -\frac{\gamma}{2T}z(z-1), \quad v = 0, \tag{2.6}$$

satisfy the equations $(2.6)_{1,2}$ and the boundary conditions $u = v = 0$ on $z = 0, l$. The third equation $(2.6)_3$ for the above u, v leads to the nonlinear equation

$$K\varphi'' - m(I_{\mathrm{I}} + I_{\mathrm{II}})\ddot{\varphi} = E(I_{\mathrm{I}} - I_{\mathrm{II}})\frac{\gamma^2}{T^2}\sin\varphi\cos\varphi. \tag{2.8}$$

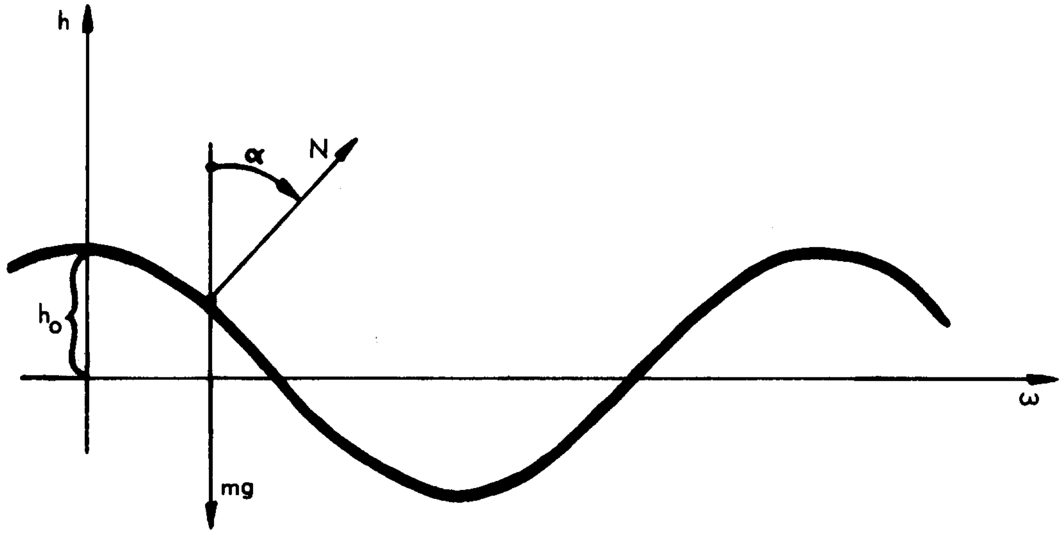

Figure 3. Statics of heavy chain on a sinusoidal surface $h = h_0 \sin \omega$.

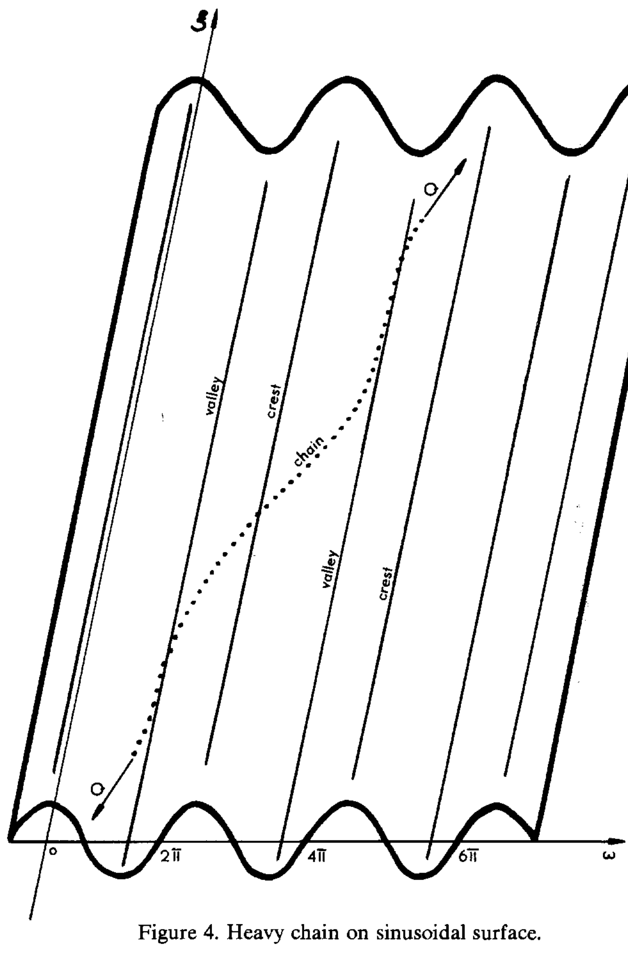

Figure 4. Heavy chain on sinusoidal surface.

In terms of the new variables

$$\xi := \sqrt{\frac{E(I_{\mathrm{I}} - I_{\mathrm{II}})}{K}\,\frac{\gamma}{T}}\,z, \qquad \tau := \sqrt{\frac{E(I_{\mathrm{I}} - I_{\mathrm{II}})}{m(I_{\mathrm{I}} + I_{\mathrm{II}})}\,\frac{\gamma}{T}}\,t, \qquad \omega = 2\varphi, \tag{2.9}$$

the equation (2.8) reads

$$\frac{\partial^2 \omega}{\partial \xi^2} - \frac{\partial^2 \omega}{\partial \tau^2} = \sin \omega. \tag{2.10}$$

There exists a large literature concerning the sine-Gordon equation (2.10). Among others this equation describes the purely geometrical problem of bending of an inextensible surface of constant negative curvature. Very important for the visualisation of the solutions in the fact that (2.10) describes the dynamics of a heavy chain on a smooth sinusoidal surface (Figs. 3 and 4),

$$h = h_0 \sin \omega. \tag{2.11}$$

3. Solutions

Introducing new variables

$$2\alpha = \xi - \tau, \qquad 2\beta = \xi + \tau, \tag{3.1}$$

the equation (2.10) may be written in the form

$$\frac{\partial^2 \omega}{\partial \alpha \partial \beta} = \sin \omega. \tag{3.2}$$

If

$$\omega_1 = \Phi(\xi, \tau) = \Phi(\alpha + \beta, -\alpha + \beta) = \Psi(\alpha, \beta), \tag{3.3}$$

is a solution, then

$$\omega_2 = \Phi\left(\frac{\xi - c\tau}{\sqrt{1 - c^2}}, \frac{\tau - c\xi}{\sqrt{1 - c^2}}\right) = \Psi\left(\frac{\alpha}{c}, c\beta\right), \tag{3.4}$$

is a solution, too. The function ω_2 is the Lie transform of ω_1. The Lie transform allows us to obtain the dynamic solution $\omega_2(\xi, \tau)$ from the static one $\omega_1 = \omega_1(\xi)$.

Another solution is

$$\omega_3 = \Phi(\tau, \xi) + \pi. \tag{3.5}$$

The function ω_3 is the π-transform of ω_1. It allows us to obtain the supersonic solution

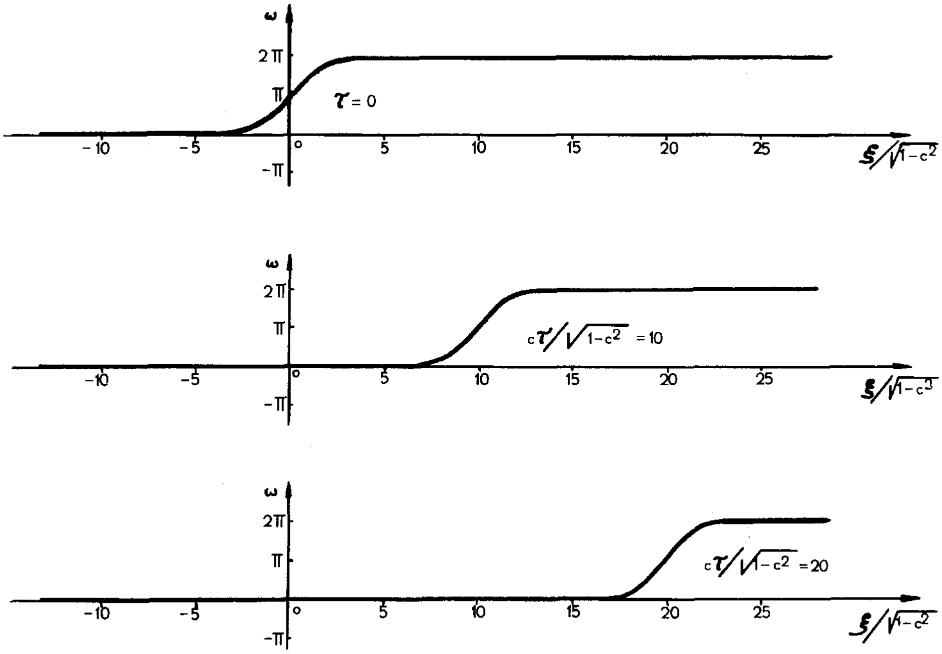

Figure 5. Solitary wave, Eqn. (3.6).

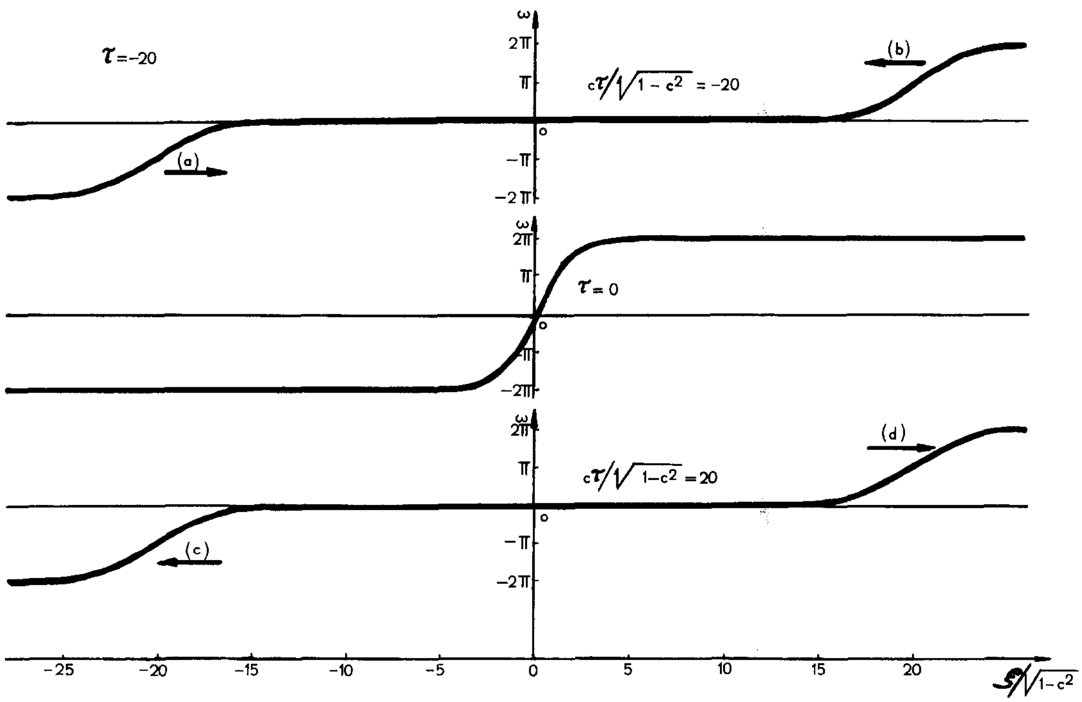

Figure 6. Collision of two solitons, Eqn. (3.7).

$\omega_2 = \Phi(c\tau - \xi)$ from the subsonic solution $\omega_1 = \Phi(\tau - c\xi)$, $c < 1$.

The system of first order partial differential equations allows us to obtain the function $\omega_4(\xi, \tau)$, i.e. the Backlund transform of ω_1. A full description of the problem is given in [1]. A large number of exact solutions to (2.10) is known. Among others there exists the solution called the solitary wave

$$\omega = 4 \operatorname{arctg} C \exp\frac{\xi - c\tau}{\sqrt{1-c^2}}, \quad C = \text{const}, c = \text{const}. \tag{3.6}$$

The solution has been shown in Fig. 5. The right end of the chain is situated in the valley $\omega = 2\pi$ and the left end in the valley $\omega = 0$. The deformed region is localised near the point $\xi = c\tau$. The distance from the supports has almost no influence on the wave profile. No force can remove in a finite time the far ends from the valleys, and in this sense the solution (3.6) preserves its identity.

The function

$$\omega = 4 \operatorname{arctg} c \frac{\cosh\dfrac{\xi}{\sqrt{1-c^2}}}{\sinh\dfrac{c\tau}{\sqrt{1-c^2}}}, \quad c = \text{const}. \tag{3.7}$$

is an exact solution, too. For three subsequent times it has been shown in Fig. 6. It represents the collision of two solitons. After the collision the waves regain their initial form. Note that in accord with $(2.9)_3$, to $\omega = 0, 2\pi$ (valleys) there corresponds $\varphi = 0, \pi$, and to $\omega = \pi$ (crest) there corresponds $\varphi = \pi/2$.

References

[1] A. Seeger, Solitons in crystals, Third International Symposium on Continuum Models of Discrete Systems, Freudenstadt (1979).

[2] A. Seeger and Z. Wesołowski, Standing wave solutions of the Enneper (sine-Gordon) equation. *Int. J. Eng. Sci.* 19 (1981) 1535–1549.